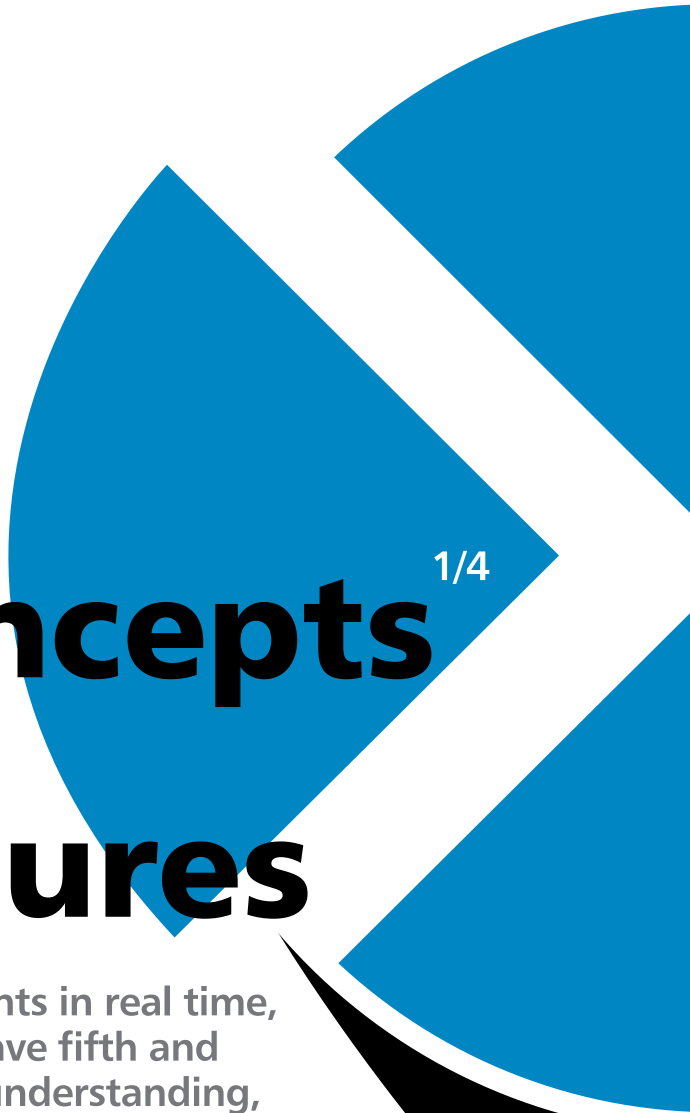
Fractions Instruction:

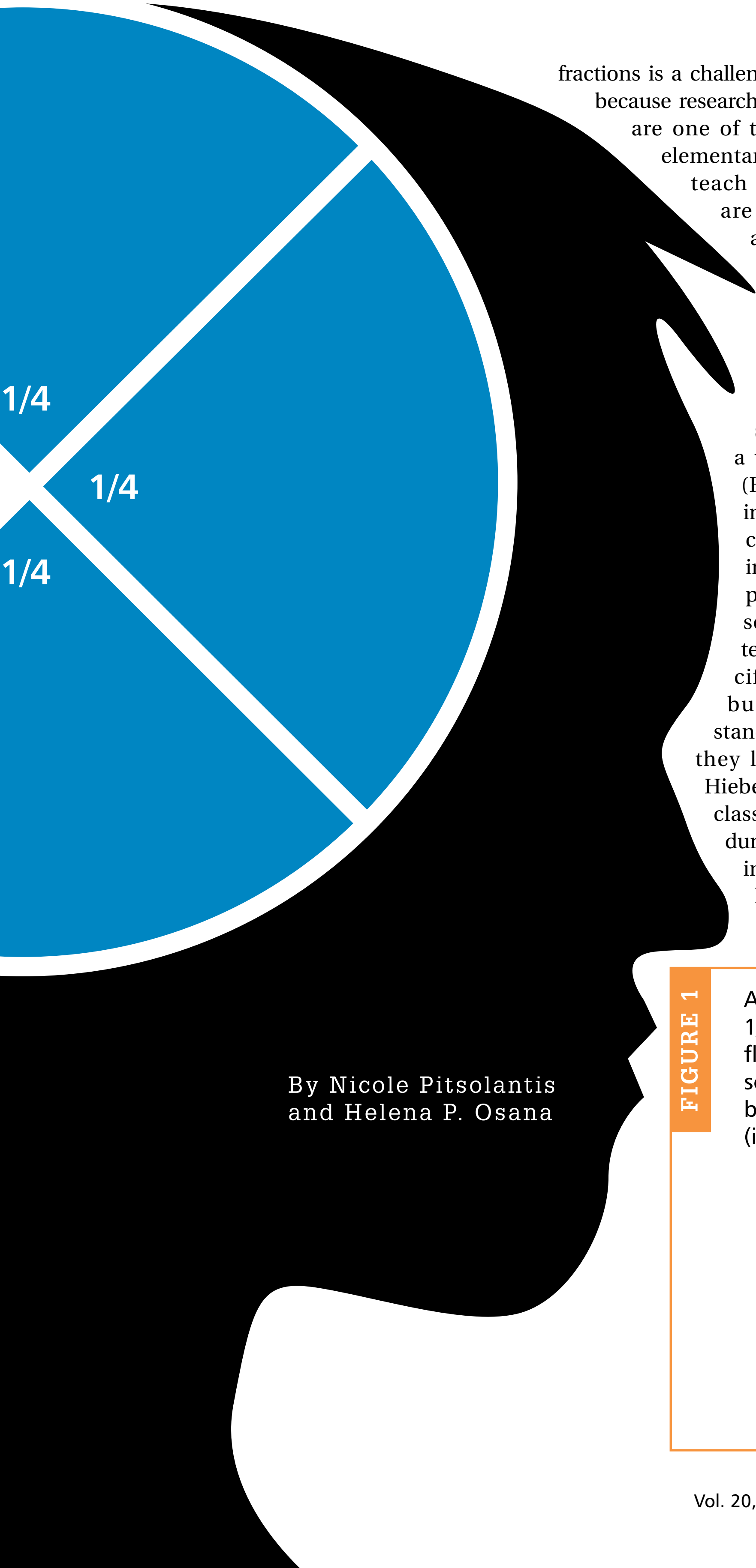
Linking Concepts and Procedures



Three specific sites, or points in real time, during problem solving gave fifth and sixth graders conceptual understanding, procedural skill, and the ability to justify their mathematical thinking about fractions.

Mark, a sixth grader, seems to be an average math student. He follows along during class discussions and always completes his work on time. During a unit on fractions, however, we asked Mark to compare 2/5 and 1/3 to determine which fraction was larger and then explain his thinking. Although Mark found the correct answer, which he illustrated with two shaded fraction strips, his flawed thinking was that fractions are "pieces" of something independent from a whole and that they can be compared as if they were whole numbers. Moreover, the images Mark called to mind when he compared fractions (comparing wholes and parts of different sizes) showed clearly that he conceptually misunderstood fractions. His ways of thinking were indicated by his difficulty making links between core concepts and the symbols and procedures used to solve fraction problems (see **fig. 1**).

Many of us have probably come across several students like Mark in our classroom, students for whom fraction concepts and procedures seem to have no link to one another and for whom meaning-making in



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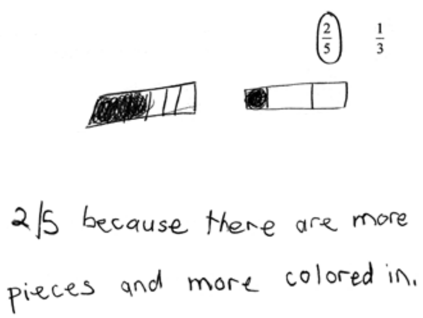
fractions is a challenge. This is not surprising, because research has shown that fractions are one of the most difficult of the elementary school math topics to teach and learn in ways that are meaningful (Mazzocco and Devlin 2008; NMAP 2008; Stafylidou and Vosniadou 2004; Wu 2008).

We describe a fractions teaching unit we used with fifth- and sixth-grade students that is based on a view of problem solving (Hiebert 1992) highlighting three phases for linking concepts and procedures in mathematics. We incorporated Hiebert's problem-solving framework into a teaching unit designed specifically to help students build meaningful understanding of the mathematics they learn in school. We use Hiebert's (1984) definitions to classify concepts and procedures as follows: *Concepts* are intuitions and ideas about how mathematics works

FIGURE 1

Although Mark's answer that 2/5 is larger than 1/3 is indeed correct, his drawings show his flawed thinking that fractions are "pieces" of something independent from a whole and can be compared in the same way as whole numbers (i.e., 5 is more than 3, and 2 is more than 1).



that make personal sense to students, who can acquire them from both everyday experiences and school instruction. *Procedures* are efficient, step-by-step rules used to solve problems. Knowing procedures involves being able to use those rules, but it also involves knowing ways to represent mathematical quantities using numbers (such as 3/4 to represent three-fourths). In this article, we call these representations *symbols*. We define *links* as connections between concepts and procedures. Making links involves tying together ideas and symbols to make mathematics meaningful. Hiebert (1992) suggested that teachers should emphasize these links during instruction so that students may "see" the underlying meanings of mathematical symbols and rules. Although procedures have their place in school mathematics, knowing why they work allows students to realize when they have made mistakes and provides the tools necessary for fixing them (Baroody 2003; NCTM 2000).

Linking concepts, symbols, and procedures

Hiebert (1992) proposed that mathematical concepts should be linked to symbols and rules during teaching to foster students' authentic, personally meaningful understanding. More specifically, he suggested three precise *sites*, or points in real time, during problem solving where such links could be especially useful to learning. At site 1, the *symbol interpretation site*, mathematical symbols are assigned meaning. Teachers can link concepts and procedures by drawing connections between symbols and anything that can give those symbols meaning. Hiebert therefore proposed that instruction should connect symbols to concrete pictures, objects, or real-world scenarios (e.g., making the link between 4/5 and four out of five people in the room).

At site 2, the *procedural execution site*, rules are used to find answers to mathematical problems. The teacher links concepts and procedures by showing students, often with pictures or manipulatives, how the rules make sense (e.g., showing why a common denominator is needed when adding fractions).

At site 3, the *solution evaluation site*, answers to mathematical problems are evaluated for their reasonableness. Teachers can relate students' written answers to either (a) real-world or concrete contexts (e.g., framing the problem in a real-world context and considering how it might be solved) or (b) their knowledge of the number system (e.g., 18/20 does not make sense as an answer to 7/8 + 11/12 = __ , because each addend is close to 1 and therefore the answer should be a number that is close to 2).

Hiebert's (1992) sites theory aligns nicely with best teaching practices in mathematics. For one, pictures and manipulatives can help children learn the meaning of the mathematical symbols they use. They can also help children think about problems and reason about ways to solve them. However, Hiebert's theory goes beyond simply presenting students with pictures and objects to understand symbols and think about problems. We now know that the best kind of learning happens when children *reflect on* what they did with their pictures or concrete objects (Sarama and Clements 2009). The power of Hiebert's theory, therefore, is that at each site, students learn how to use mathematical concepts to reflect on the symbols and procedures they used to solve problems. By making clear links among concepts, symbols, and procedures, which is at the heart of the theory, the teacher fosters the kinds of reflections that make problem solving meaningful.

We used Hiebert's (1992) ideas to create a unit on fractions for fifth and sixth graders and found that the teaching we presented helped students learn about fraction concepts and procedures and make the kinds of links that we argue are so critical for mathematical understanding (Osana and Pitsolantis in press). Below we describe in detail the particular teaching activities we used, and we illustrate how they helped students improve their prior thinking. This teaching method is designed to take the learner through the entire problem-solving sequence, beginning with the interpretation of mathematical symbols (site 1), followed by the use of the procedures (site 2), and through to the evaluation of the solution (site 3). Teachers, therefore, must make explicit the links between concepts and procedures at each of these three points while students are working on problems in class.

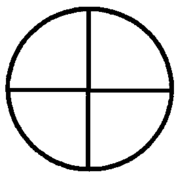
Site 1: Links between fraction concepts and symbols

Our teaching unit focused on the part-whole meaning of fractions; we emphasized that a

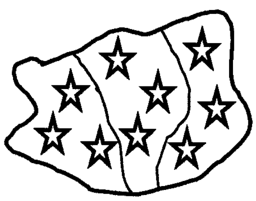
FIGURE 2

To illustrate the concepts of *whole* and equal partitioning (that the quantity expressed by a fraction relates to one or more equal parts of the whole), the teachers presented various models.

(a) In this region model, the circle is the whole.

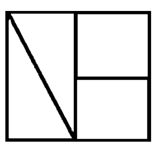


(b) The whole is the entire set of objects.



(c) This is a length model.

(d) This is an area model.



whole of some kind is partitioned into equal parts and a certain number of those parts is being considered. To show various representations of the part-whole meaning to our students, we used several models: the region, set, length, and area models (see **fig. 2**; for the purpose of this unit, *models* are diagrams or other depictions that show what fractions are). We began the unit by teaching the concept of the *whole*; students must come to know that fractions represent *parts* in relation to a whole. We therefore suggest spending ample lesson time allowing students to construct meaningful understanding of the concept of the *whole* and the idea of equal partitioning: that the quantity expressed by a fraction relates to one or more equal parts of the whole.

To illustrate these ideas concretely, we began by presenting students with various models (see **fig. 2**) and discussing what the whole was in each. We also discussed the partitioning of each whole, explicitly pointing out that in each model, all parts are equivalent. Next, we practiced naming each part by saying, "This is one equal part of the whole," and then counting the parts. Using the example of **figure 2a**, for instance, the teacher would ask the following questions:

- "What is the whole?" (the circle)
- "How many equal parts are in the whole?" (four)
- "What do we call each part of a whole that is partitioned into four equal parts?" (fourths)
- "Let's count these parts." (one-fourth, two-fourths, three-fourths, four-fourths)

We always summarized these discussions by directly stating that when we talk about fractions, we are talking about amounts that are equal parts of a whole thing. In this way, our teaching explicitly linked the models used to represent fractions with the fundamental concepts of part-whole and equal partitioning.

After teaching these ideas, we moved to the critical aspect of linking these concepts to ways of representing fractions with symbols. As mentioned above, teaching at site 1 involves making connections between symbols and concrete objects or real-world situations. We used drawings and paper models as concrete representations. Examples of our connections to the real world included referring to a rectangular region as a candy bar or to a line segment as the length of one's shoelace. After naming the whole and equally partitioning it, we shaded the specified number of parts to show the fraction that we were discussing. For example, for the quantity

four-sixths, we shaded four of six equal parts of the whole. Because students had already practiced partitioning, naming, and counting parts, they were able to see the parts and the whole that together represented four-sixths. At this point, we introduced the fraction symbol 4/6 and directly linked it to the concrete model by showing the symbol and the model side by side (see **fig. 3**). We explained that the symbol means four of six equal parts, that the 6 below the line refers to the whole that has been partitioned into six equal parts, and that the 4 above the line refers to the number of equal parts, in this case 4, that has been selected.

Linking symbols to models that captured basic fraction concepts was central to our teaching because we discovered that before the unit began, many students could neither explain,nor show with a picture the quantity associated with fraction symbols that they could read and write. For example, one student, Padma, could use the correct symbol to represent a fraction written in words but could not explain the concepts behind the symbols she wrote (see **fig. 4a**). By relating fraction symbols to concrete objects and real-world scenarios, we were able to highlight their meaning and help students like Padma come to understand the symbols they used. After we taught the unit, Padma showed a much better understanding of fraction symbols (see **fig. 4b**). That is, she was able to explain what the numbers meant in terms of the part-whole meaning of fractions.

Site 2: Links between fractions procedures and underlying rationales

As stated above, concepts and procedures are linked at site 2 when teachers use concepts to explain why mathematical procedures work. We therefore frequently made direct links between symbolic procedures and related concepts by creating meaning for those procedures. We did this through the use of concrete and pictorial models as well as through real-world story problems. Fractions can be compared, added, and subtracted only if the wholes are the same size; therefore, all our teaching of procedures involved the use of same-size wholes as examples. Furthermore, all problems that students worked on involved same-size wholes. When we posed problems that involved only symbols (i.e., with no accompanying pictures or models),

FIGURE 3

Teachers directly linked the symbol $\frac{4}{6}$ to the concrete model by showing them side by side and explaining why this symbol means four of six equal parts.

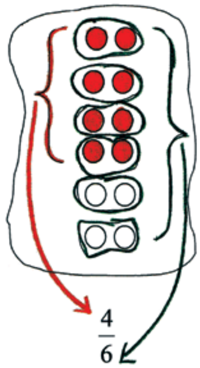
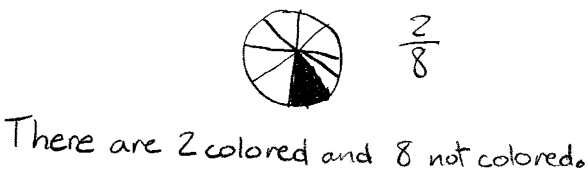


FIGURE 4

Before the unit began, many students who could read and write fractions could neither explain nor show with a picture the quantity associated with fraction symbols.

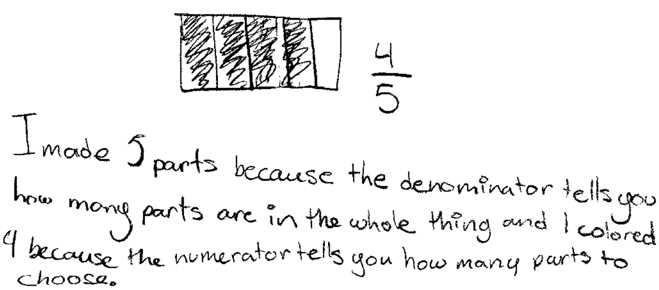
(a) Padma could use the correct symbol to represent a fraction written in words but could not explain the concepts behind the symbols she wrote.

Write the fraction for the number two-eighths, and draw a picture to explain your thinking.



(b) After being taught the unit, Padma could explain what the numbers mean in terms of the part-whole meaning of fractions.

Write the fraction for the number four-fifths, and draw a picture to explain your thinking.



we reinforced the idea that the wholes were the same size.

To illustrate how procedures are linked to their rationales at site 2, we use an example from our lesson on addition with fractions. Before we taught the unit, we posed a problem to students that involved adding 2/6 and 3/6. As Michelle's work shows (see **fig. 5a**), she made the error of adding across both the numerators and the denominators to get her answer of 5/12. Her explanation and her symbolic procedure point to a lack of understanding of both fraction concepts and the procedures used to add fractions. Michelle's reasoning is common and probably stems from misapplying whole-number knowledge to addition with fractions (Ni and Zhou 2005; NMAP 2008).)

A frequently taught rule for adding fractions with common denominators is to add the numerators but not the denominators. To make the procedure meaningful for students, we did the following. We began by revisiting the part-whole concept that we had discussed during earlier lessons. Using both paper models and drawings of fractions with same-size wholes, we reviewed the idea that a fraction is a quantity that represents a part of a whole and in this way is different from a whole number. We once again linked fraction symbols to drawings and paper models by showing the symbol and model side by side, thus also revisiting the idea that the way fraction numbers are written (i.e., 3/6) also refers to a part of a whole. Next, using the same models, we reviewed counting fractional parts so that we could move from *counting* parts to *adding* them. For example, for the fraction 3/6, one can count, "One-sixth, two-sixths, three-sixths," or alternatively, one can say, "One-sixth and one more sixth equals two-sixths," and so forth, to illustrate adding fractional quantities. Using the counting strategy to add fractional parts makes clear the idea that *sixths* are being added, and not whole numbers or any other fractional part (i.e., such as twelfths in the context of Michelle's solution).

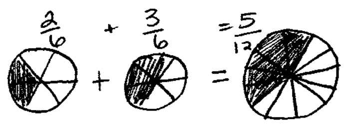
After reviewing part-whole concepts and naming and counting fractional parts, we demonstrated the procedure and linked it to models that illustrate its meaning (see **fig. 5b**, and note the use of drawings depicting the same-size wholes). The connections we made between the symbolic procedure and the models allowed us to explicitly demonstrate that sixths were being added, that the numerator represents the portion of the whole that is being considered, and therefore, the portion itself is the amount of interest. This link highlighted *why* we add only the numerators and not the denominators in this context. After we taught the unit, Michelle demonstrated a better understanding of this addition procedure when she used it to solve a similar problem (see **fig. 6**).

FIGURE 5

Michelle's initial reasoning is common and probably stems from misapplying whole-number knowledge to addition with fractions (Ni and Zhou 2005).

(a) Her explanation and symbolic procedure pointed to a lack of understanding of both fraction concepts and the procedures used to add fractions.

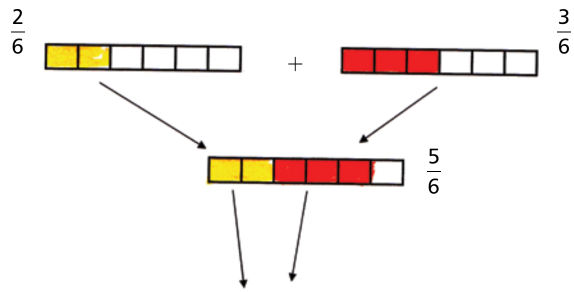
Alexa ate two-sixths of a chocolate cake, and Steven ate three-sixth of a strawberry cake. How much of a cake was eaten altogether?



We have to add them up so she ate 2 and he ate 3 so that's $\frac{5}{12}$ because I combined the cakes because it's 12 cakes all together when you add them.

(b) The connections between the symbolic procedure and the models allowed the teachers to explicitly demonstrate that (1) sixths were being added; (2) the numerator represented the portion of the whole being considered; and (3) the portion itself was the amount of interest.

Problem: $\frac{2}{6}+\frac{3}{6}=$



$$\frac{2}{6}+\frac{3}{6}=\frac{2+3}{6}=\frac{5}{6}$$

FIGURE 6

After the unit had been taught, Michelle solved a similar problem, demonstrating a better understanding of the addition procedure. She now clearly understood that (1) fifths were being added; (2) the size of the whole did not change after adding the parts; and (3) adding the denominators would result in 4/10, a quantity that would not make sense.

$\frac{3}{5} + \frac{1}{5} = \frac{4}{5}$

It's $\frac{4}{5}$ because you don't add the denominators because it's really fifths you add. You need 5 to make the whole pizza so if it would be $\frac{4}{10}$ it would be a different amount.

Site 3: Links between symbolic solutions and their reasonableness

One important aspect of understanding in mathematics is the ability to judge the reasonableness of an answer. In other words, students must be able to tell whether their answers, often expressed in symbols, make sense. When they can do so, they are more likely to notice that answers obtained by applying faulty procedures are not reasonable. Hiebert (1984) referred to the judgment of solutions as testing answers, from both a mathematical and real-world perspective, to see if they make sense. The general idea is to have children think about their answers in terms of the skills, knowledge, and concepts they already have, so they can figure out if their answers are sensible. For students to better evaluate their solutions, therefore, our teaching focused on linking their answers to either (a) real-world or concrete contexts or (b) their knowledge of the number system.

We illustrate with an example from one lesson on comparing and ordering fractions (see **fig. 7a**), which shows a typical order problem involving the fractions 5/6, 3/4, and 2/3 and the solution that one student, Jordan, gave before the unit. Jordan's explanation showed that he was thinking of the numerators and denominators of each fraction as separate numbers and, furthermore, that the numerators do not play a role in determining the order of the fractions. Instead, Jordan believed that to solve the problem, he should compare the denominators alone one to the other, as one would compare whole numbers. In this particular example, Jordan ended up with the right order by applying an incorrect procedure because the numbers in the problem happened to allow for it. However, this type of faulty thinking would not be helpful to Jordan when solving similar problems with different numbers. To encourage him and the others to reflect on the reasonableness of their answers, we linked the problem to a real-world context. For example, we asked students to think about the problem in the following manner:

> You and two friends are each eating a cookie that is the same size. One friend ate 5/6 of his cookie, your other friend ate 3/4 of his cookie, and you ate 2/3 of your cookie. Order the fractions to show who ate the least to the most.

Couching the problem in this context allowed students to think about it in a way that was personally meaningful and to consider the numbers in the problem as real objects. As one student put it, "If you draw the cookies, you can see who ate the most and who ate the least." After instruction, Jordan, too, was clearly able to evaluate his solution to similar problems more sensibly. He learned to think about fractions as different from whole numbers and to call on the number knowledge he already had—percentages, in this case—to justify his answer (see **fig. 7b**).

At any time during the unit when an answer was presented to the class, either by the teacher or by students, its reasonableness was checked. At such points, links at site 3 were highlighted, again by either stating the problem in a concrete or story context or by applying what we knew about numbers to judge the correctness of the answer. For example, during one lesson on comparing fractions, a student named Juan initially answered that 2/6 was larger than 1/2 when called upon to share his solution. We prompted him to connect the problem to a real-world scenario that was personally meaningful to him. We asked, "If these were two cakes of the same size, would you rather have half of one or two-sixths of the other?"

Juan immediately changed his answer: "One-half because it's half, it's more. Oh, now I get it!"

Similarly, during student work periods, the teacher circulated around the classroom and either introduced concrete or story contexts to have students think about their answers or

asked them to use their number knowledge to evaluate their work. During one such work period, for instance, Dilara was asked to justify her solution to a problem that involved converting the mixed number 2 1/4 to an improper fraction. Dilara applied the standard procedure of multiplying the whole number by the denominator ($2 \times 4 = 8$), then adding the numerator ($8 + 1 = 9$), and stating that number over the original denominator (fourths) to get the answer 9/4. When asked if she could draw a picture to test the reasonableness of her answer, Dilara drew three circles of the same size and replied, "It's fourths, so if I put four parts in each of these circles and color in nine, I'll have two whole ones shaded and then 1/4 of one shaded; so it's right."

An instructional method for all

Our teaching was based on a theory of general problem solving in mathematics that we tailored for use in our unit on fractions. We found this method to be effective in that our students showed a more meaningful understanding of fractions after the unit. We propose that this method can be used with most, if not all, mathematical topics, and likely at any grade level. The ideas we outlined above for making direct connections between concepts and procedures at specific points in time can be applied to any topic, with appropriate changes. As such, we believe this method to be a useful one for teachers because it is flexible and wide-ranging.

FIGURE 7

Before Jordan studied the unit, his explanation of how to order fractions showed that he thought of numerators and denominators of each fraction as separate numbers.

(a) Jordan also had a mistaken notion that numerators do not play a role in determining the order of the fractions.

Order these fractions from smallest to largest: $\frac{5}{6}, \frac{3}{4}, \frac{2}{3}$

$\frac{2}{3}, \frac{3}{4}, \frac{5}{6}$

This is my answer because 3 is the smallest number and then 4 and then 6. I looked at the bottom numbers and put them in order from smallest to largest. The bottom number tells the order.

(b) To understand fractions and justify his answer, Jordan learned to call on his prior number knowledge of percentages.

Order these fractions from smallest to largest: $\frac{3}{4}, \frac{2}{3}, \frac{2}{6}$

$\frac{2}{6}, \frac{2}{3}, \frac{3}{4}$

3/4 is 75% so it's the biggest. 2/6 is equal to 1/3 and that's 33% so it's the smallest because 2/3 is double that so it's 66%. And if I draw it you can see that it's true.

Common Core Connections

3.NF.A.3d SMP 1
4.NF.A.2 SMP 3
6.RP.A.3

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